BAYESIAN PROCEDURE IN SEQUENTIAL SAMPLING

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FOREWORD

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SUMMARY

The research is devoted to the task of modifying the Bayesian sequential sampling procedure developed by S. G. Gilbreath [8] in 1966. The procedure considers a prior distribution for the process quality and makes use of inspection costs plus decision losses as the measure of effectiveness. Bayesian decision rules relative to chosen prior probability distributions are the bases for computing decision losses.

The following alternatives are to be considered to sentence a lot at any point in the course of inspection:

(1) accept the lot now,

(2) reject the lot now,

(3) inspect one more item and accept or reject the lot, and

(4) inspect two more items and accept or reject the lot.

The costs and expected losses for the four alternative decisions are:

(1) $K_a \, E(y|x_n)$

(2) $K_r[(N-n) - E(y|x_n)]$

(3) $$K_f + K_s + E(y|x_{n+1})\left[1 - \frac{E(y|x_n)}{N-n}\right] + K_r[(N-n-1) - E(y|x+1_{n+1})] \cdot \frac{E(y|x_n)}{N-n} \qquad \text{for } n = 0 \qquad \text{or}$$

$$K_s + E(y|x_{n+1})\left[1 - \frac{E(y|x_n)}{N-n}\right] + K_r[(N-n-1) - E(y|x+1_{n+1})] \cdot \frac{E(y|x_n)}{N-n} \qquad \text{for } n = (1, 2, \ldots\ldots, N)$$

$$(4)\ K_f + 2K_s + E(y|x_{n+2})\left[1 - \frac{E(y|x_{n+2})}{N-n}\right]\left[1 - \frac{E(y|x_{n+2})}{N-n-1}\right]$$

$$+ K_r[(N-n-1) - E(y|x+1_{n+1})]\left[2\left\{1 - \frac{E(y|x+1_{n+2})}{N-n}\right\}\right]\left[\frac{E(y|x+1_{n+2})}{N-n-1}\right]$$

$$+ K_r[(N-n-2) - E(y|x+2_{n+2})]\left[\frac{E(y|x+2_{n+2})\left\{E(y|x+2_{n+2}) - 1\right\}}{(N-n)(N-n-1)}\right] \qquad \text{for } n = 0$$

or

$$2K_s + E(y|x_{n+2})\left[1 - \frac{E(y|x_{n+2})}{N-n}\right]\left[1 - \frac{E(y|x_{n+2})}{N-n-1}\right]$$

$$+ K_r[(N-n-1) - E(y|x+1_{n+2})]\left[2\left\{1 - \frac{E(y|x+1_{n+2})}{N-n}\right\}\right]\left[\frac{E(y|x+1_{n+2})}{N-n-1}\right]$$

$$+ K_r[(N-n-2) - E(y|x+2_{n+2})]\left[\frac{E(y|x+2_{n+2})\left\{E(y|x+2_{n+2}) - 1\right\}}{(N-n)(N-n-1)}\right]$$

for $n = (1, 2, 3, \ldots\ldots, N)$

where

N = lot size

n = sample size

x = cumulative number of defective items in accumulated sample

y = the number of defectives in the uninspected portion of the lot

K_f = ratio of the fixed sampling cost to the decision loss accompanying acceptance of a defective item

K_s = ratio of the variable sampling cost per item inspected to the decision loss accompanying acceptance of a defective item

K_r = ratio of the decision loss resulting from rejection of a good item to the decision loss accompanying acceptance of a defective item

$E(y|x_n)$ = the expected number of defectives in the uninspected portion of the lot given x defectives in a sample of size n drawn from the lot

At any value of n if (1) or (2) becomes cheaper than (3) and (4), inspection is stopped and the lot is sentenced; otherwise, one more item is inspected and the procedure is iterated until it is economical to sentence the lot without inspecting one additional item.

A mixed binomial prior distribution is used in developing the model and is applied to an hypothetical example to demonstrate the effectiveness of this model. The iterative procedure is programmed for a Burroughs B-5500 electronic digital computer and the model is compared with Hald's optimum single sampling plan and Gilbreath's sequential sampling plan. The response of the modified model to shifts in statistical and economic parameters is tested.

The modified sequential sampling procedure is comparable to Gilbreath's sequential procedure. The average unit costs are not significantly different from Gilbreath's procedure. Average sample sizes are of the same order in both, Gilbreath's sequential procedure and the modified sequential procedure. The modified sequential procedure is economically superior to accepting lots without inspection.

CHAPTER I

INTRODUCTION

The General Problem

This study is an investigation of the problem of item by item sequential sampling plans which are designed on the basis of a specific prior distribution and certain cost and losses. The costs consist of fixed and variable inspection costs, and the losses are due to either accepting a bad item or rejecting a good item. The determination of the decision losses is based upon prior knowledge of the lot quality which may be a guess. In general, the problem is limited to a certain type of manufacturing situation which has the following characteristics [8]:

a. the procedure will be applicable to item by item sequential sampling for attributes,

b. the study is limited to production inspection where inspection is performed for the purpose of accepting lots of product containing few defectives and rejecting those showing indication of containing many,

c. the process responsible for the quality characteristics being inspected is stable to the degree that a quality process curve or prior distribution can exist.

Background

In recent years, prior distributions and Bayes' theorem have occupied a prominent place in the literature of sampling inspection. A.

Hald [7] in 1960 presented a model for designing an optimum single sampling plan using as criteria costs and the prior distribution of process quality. Using the same approach, Gilbreath [8] in 1966 proposed a procedure for item by item sequential sampling for attributes. But Gilbreath's procedure did not prove to be as economical as Hald's single sampling procedure. In his work, Gilbreath recommended continued investigation with a goal of finding a Bayesian sequential procedure economically superior to Hald's optimum single sampling plan. This research is a part of that continued investigation.

The Specific Problem

There are a few factors which must be investigated carefully. First, the determination of an item by item sequential sampling plan requires a probabilistic approach since samples are drawn at random from a lot which is randomly selected from a production process. Second, there are statistical and economic parameters which need to be estimated. These parameters are the process fraction defectives p_1, p_2 and proportion of total production from each process level w_1, w_2 in case of a two-point mixed binomial prior distribution; the fixed sampling cost C_f; the variable sampling cost per item inspected C_s; the decision loss accompanying acceptance of a defective item C_a; and the decision loss resulting from rejection of a good item C_r. We assume that, for the purpose of this research, reliable estimates of the statistical and economic parameters are available. The assumptions are important because the effectiveness of the model depends upon the reliability of these estimates. Gilbreath [8] in his work proposed a conceptual model for identifying economic criteria.

The Study Procedure

Gilbreath's [8] item by item sequential sampling model designed on the basis of prior distributions and costs will be modified. The modified model will be developed for a mixed binomial prior distribution and programmed for the Burroughs B-5500 electronic digital computer. The model will be evaluated for the two-point mixed binomial prior distribution at different values of the process curve and cost parameters. The effectiveness of the model will be compared to Gilbreath's item by item sequential sampling plan and Hald's [7] optimum single sampling plan. The behavior of the model for various values of process fraction defectives p_1, p_2 and cost parameters C_s, C_r will be investigated.

CHAPTER II

LITERATURE SURVEY

In recent years there are many publications which advocate the use of prior distributions and costs as the design criteria for sampling inspection plans. Johnson [1] surveyed the literature concerning sampling procedures based upon economic and noneconomic criteria and concluded that Bayes' principle furnishes the best criterion for acceptance inspection decisions. Pfanzagl [2] in her paper wrote,

> The great number of papers dealing with the problem of sampling procedures based upon prior distributions and costs suggest a general dissatisfaction with the usual methods of selecting sampling according to their O.C. curves without any consideration of costs and without any information about the prior distribution of fraction of defectives in different lots except for average fraction of defectives.

Barnard [3] in 1954 pointed out the close correspondence between the theory of statistical decisions and statistical inspection. He wrote that alternate courses of action, acceptance of a batch, rejection of a batch, rejection or acceptance subject to further inspection are fairly clear out, and quite good estimates can often be made of monetary losses arising from wrong decisions. He considered the two-component mixed binomial prior distribution to be evident for small lot sizes and suggested that more components should be introduced as the lot size increases.

Much work has been done in the design of single sampling plans based on the criteria of costs and prior distributions and little has been done towards design of item by item sequential sampling plans based on the same criteria. It is worthwhile to review the literature of Bayesian theory as applied to inspection decision making and single sampling procedures based on economic and prior distribution of process quality criteria.

Bayesian Theory in Decision Making

Locke [4] introduces the concept of a Bayesian approach to statistical decision making. He differentiates between the Bayesian approach and the classical approach by saying that classical methods do not consider any prior representation of the probability of occurrence of the parameter values while Bayesian methods make use of a prior probability distribution of the unknown parameter to provide weights for the conditional error and losses. In the Bayesian approach, for each decision rule an unconditional expected loss is computed and that decision which gives a minimum loss is considered. He presents an analysis to determine optimal sample size which minimizes the opportunity loss associated with sample size plus sampling cost. The expected opportunity loss is computed with knowledge of the loss function and prior probability distribution of lot quality. The author believes that the greatest usefulness of the analysis is to force the decision maker to give careful consideration to the risks associated with the decision and that it provides greater insight into the entire decision process. J. Cornfield [5] wrote another paper on Bayes' theorem. This paper explains Bayes' theorem and the problems that arise in applying it to practice. He gives an account of the use of the loss function in decision theory and presents an expression for average loss $\overline{L}(d, \theta)$ for any decision rule d, making use of Bayes' theorem. He also uses prior probability to compute average loss for all possible states of nature. He presents a rule, "Act so as to minimize the average loss," which is defined as a Bayesian decision rule relative to the chosen prior probability. The paper indicates that probabilities need to be relative frequencies and prior probabilities cannot be assigned from experience or from principles but should be a degree of belief. Thus the subjective view is regarded as inescapable. Schlaifer [6] gives a detailed coverage of Bayesian statistics.

Single, Double, and Multiple Sampling

Hald [7] in 1960 based sampling inspection directly on the hypergeometric distribution instead of the usual approximations. He formulated a model for single sampling considering a prior distribution and using cost as the measure of effectiveness. Considering $P(X)$ the prior distribution of defectives in a lot of size N, Hald related the probability of x defectives in a sample of size n to the probability of X defectives in the lot from which the sample is drawn. From this he derives the marginal distribution of x in the sample as a function of X, n, N, and $P(X)$. Hald's single sampling inspection procedure consists of minimizing the total cost function

$$K(n, c) = n(K_s - K_r) + N\ K_r\ (N-n) + (P_n(x) - K_r)\ g_n(x)$$

where

$$K_r(\text{sorting}) = \frac{\text{average cost per item}}{\text{average cost of accepting a defective item}}$$

$$K_r(\text{nonsorting}) = \frac{\text{manufacturing cost per item}}{\text{average cost of accepting a defective item}}$$

$$K_s = \frac{\text{sampling and testing cost per item inspected}}{\text{average cost of accepting a defective item}}$$

$g_n(x)$ = marginal distribution of x

$K(n, c)$ = total expected unit cost

He evaluated the model for the rectangular, Polya, hypergeometric and mixed binomial distributions, and a wide range of cost parameters. The results appear in tabular form. Nothing is mentioned concerning determination of cost parameters. This model is universally applicable to single sampling and is a practical step towards reality. At the end of his work, Hald recommends that his approach may be extended to double and sequential sampling. Later in 1966 Gilbreath [8] extended Hald's approach to item by item sequential sampling. Hald [9] published another paper recently, in which he gives a comprehensive theory of sampling inspection for attributes based on producer's and consumer's risks. He considered the average loss to be a linear combination of producer's and consumer's risks and based on decision losses and a two-point mixed binomial prior distribution, developed a standard average cost formula

$$R(N, n, c) = n + (N-n)\ (r_1\ Q(p_1) + r_2\ P(p_2))$$

where

$r_1 = w_1(K_r(p_1) - K_a(p_1))\ (K_s - K_m)$

$r_2 = w_2(K_a(p_2) - K_r(p_2))\ (K_s - K_m)$

$Q(p_1)$ = producer's risk

$Q(p_2)$ = consumer's risk

p_1, p_2 = quality levels for two-component mixed binomial distribution

w_1, w_2 = weighting factors for two-component mixed binomial distribution

$K_a(p)$, $K_r(p)$, $K_s(p)$ = cost functions

K_a, K_r, K_s = average costs

Hald's single sampling plan consists of determining value of (n, c), minimizing $R(N, n, c)$, and using this sampling plan if min R is less than the cost of accepting or rejecting all lots without inspection. He formulated minimum average costs for different producer's and consumer's risk and concluded that a system with a fixed producer's risk may be expected to give low efficiencies for large lots but is reasonable for small lots. At the end of his work Hald recommends that his model may be generalized by introducing a polynomial cost function or a more general prior distribution. Hald gives a very interesting approach as it is based on basic concepts of sampling inspection.

Guthrie and Johns [12] developed a Bayes' acceptance procedure for large lots. Their cost model considers acceptance and rejection of the uninspected portion of a lot on the basis of the sample drawn. They developed an optimum procedure in the Bayesian sense for various classes of a-priori probability distributions.

Smith [10] considers a minimum loss approach in designing a single sampling plan. He considers the following cost function:

$$\text{total cost of acceptance} = a_1(N-n) + a_2(X-x) + S_1 n + S_2 x$$

$$\text{total cost of rejection} = r_1(N-n) + r_2(X-x) + S_1 n + S_2 x$$

where

a_1 = cost of accepting an item without regard to quality

a_2 = additional cost if an accepted item is defective

r_1 = cost due to rejecting an item without regard to quality

r_2 = additional cost if a rejected item is good

s_1 = cost of inspecting an item without regard to quality

s_2 = additional cost if an inspected item is defective

Loss depends upon the inspection plan (n, c) and the number of defectives in the lot (X). Considering the loss associated with a particular sampling plan $L(X, n, c)$ and probability of having X defectives in a lot of size N, Smith derives a Bayes' loss function $L(P_n(X), n, c)$. Using a beta prior distribution he arrives at a relation for optimum c and n. The approach is based on data which can easily be obtained.

Wurtele [11] uses a Bayesian procedure and costs for rectifying inspection. She develops a cost model in which the last item should be inspected if, and only if

$$Z(d, N-1) > Z(d, N) = K\ N$$

where

d = cumulative number of defectives found

N = the lot size

K = cost of inspecting one item

$Z(d, n)$ = expected value of risk when d defectives have been found in a sample of size n

If N-2 items have been inspected and d defectives found, the expected value of the risk is

$$\mathrm{Min}(Z(d, N-1),\ K(N))\ (1-E(p|d, N-2))$$
$$+\ \mathrm{Min}(Z(d+1,\ N-1),\ K(N))\ (E(p|d, N-2))$$

where

$E(p|d,n)$ = fraction defective in the uninspected portion of the lot when d defectives have been found in a sample of size n

This quantity is compared with $Z(d, N-2)$ to find whether to terminate inspection at $(d, N-2)$ and the procedure continues until comparison is made for all points (d, n) and all the stopping points are found. These points describe either a single sampling or multiple sampling plan. The procedure applies to cases when there are only two decisions, accept or continue sampling and when the variable cost of sample inspection is equal to the variable cost of screening inspection. This procedure does not apply to fixed cost of sampling inspection.

Pfanzagl [2] based her paper on Hald's [7] results. She analyzed the effect of small changes in the prior distribution on Hald's optimum single and double sampling plans. The paper indicates that there is only a moderate influence of the prior distribution on the optimum single and double sampling plans. The paper does not deal with the effect of small changes in costs on the optimal procedure. Pfanzagl used the Polya distribution in her analysis.

Sequential Sampling

The most original work in sequential sampling is by Wald [13]. Wald designed the item by item sequential sampling plan as a probability ratio test. He gave no consideration to optimality of cost in his procedure.

Champernowne [14] deals in his paper with the problem with determining optimum sequential schemes which minimize the sum of the decision and inspection costs. He writes in the introduction to his paper:

> In order to choose the most economical sequential sampling procedure prior knowledge is required. The prior knowledge is concerned with
>
> (1) the average quality of the batches to be tested and the variation between batches of quality about that average
>
> (2) the cost of inspection and its dependence on the amount of inspection undertaken, and
>
> (3) the cost involved by deciding wrongly to accept or wrongly to reject a batch, and the way this cost depends on the quality of the batch.

Champernowne found the most economical scheme by minimizing the total expected cost and taking into account both (1) the cost of testing and (2) decision costs, i.e., cost due to wrongly accepting or rejecting a lot. Then he measured the efficiency of the recommended scheme and found it to have a high efficiency. Prior distributions are not used in this approach.

Breakwell [15] and [16] wrote two papers in which he uses inspection costs and decision losses as criteria and the minimax principle of choice in designing sequential sampling plans. In his first paper he uses the normal approximation to design sequential tests when the acceptable fraction defective is not very small. In the latter he uses Poisson approximation to design sequential tests for very small fraction defective. He defines his minimax criteria as the procedure for which the risk function, maximized with respect to some unknown fraction p shall be as small as possible. The lot fraction defective will be least favorable to the producer. Gilbreath [8] in his work comments on this approach:

> This seems unduly severe as a general condition. If no information is available concerning the form of the distribution of lot frac-

tion defective, use of the minimax principle while the prior distribution is being analyzed may be desirable because of its conservative nature. However, a more realistic approach should be followed as the information about the form of the process curve is made available.

Vagholkar and Wetherill [17] designed a procedure for determining the most economical binomial sequential probability ratio test. The analysis is based on a two-component mixed binomial prior distribution.

Gilbreath [8] in 1966 extended Hald's optimum single sampling plan to an item by item sequential sampling plan for attributes. His proposal considered the prior distribution of process quality, inspection costs, and decision losses. He gave models for the hypergeometric, Polya, binomial, and mixed binomial prior distributions. At any point during inspection he considered the following alternatives:

(1) accept now,

(2) reject now, and

(3) inspect one more item and accept or reject.

The sequential decision rule requires termination of inspection at any point for which the expected decision loss is less than the cost of inspecting one additional item plus the expected decision loss if a decision is made at that point. The inspection continues so long as it is economical to make a decision following inspection of one additional item. Cost parameters are the same as used in Hald's work [7]. Analysis is complete for the two-component mixed binomial prior distribution. Gilbreath programmed the model for the Burroughs 220 electronic digital computer and analyzed the model for various combinations of statistical and cost parameters. He concluded that Hald's optimum single sampling plan is economically superior to the proposed item by item sequential sampling plan but that the magnitude of the economical disadvantage of his proposed method is small. He considered the early sentencing, resulting after the examination of excessively small samples, as the major cause for the inferiority of his proposed procedure. An advantage of his method is that it is adaptable to inspection procedures using on line access data processing equipment. At the end of his work, Gilbreath recommends extension of his proposed method, with the goal of finding a Bayesian sequential procedure that is economically superior to Hald's optimum single sampling plans.

CHAPTER III

THE COST MODEL FOR SEQUENTIAL SAMPLING

Prior Distribution of Lot Quality

Sampling inspection procedure can be described as follows. Lots of N units are obtained from some manufacturing process and are inspected by sampling inspection. The number of defective items, X, in such lots generate an a-priori distribution of some form. The lots contain (N-X) good items. Sampling divides a lot into a sample of n items and an uninspected portion of (N-n) items. Inspection reveals x defectives in the sample, leaving y = (X-x) defectives in the remainder of the lot. Assuming simple random sampling without replacement, the conditional distribution of x, given X, is hypergeometric with parameters X, N, and n. For a given distribution of X and sampling without replacement, some important properties of x and y may be shown [7].

The following analysis from Hald appears in the same form in Johnson's [1] and Gilbreath's [8] work.

The distribution of the number of defectives in the lot will be denoted by $h_N(X)$, where $X = 0, 1, 2, \ldots\ldots, N$. The lot fraction defective is X/N.

Sampling Distribution

The conditional distribution of x, when there are X defectives in the lot is

$$f(x|X) = \begin{cases} \binom{X}{x}\binom{N-X}{n-x}\Big/\binom{N}{n}, & x = \max(0, n-N+x),\ 1, \ldots., \min(n, x) \\ 0, \text{ otherwise} \end{cases} \tag{3-1}$$

Equation (3-1) may be written in the following equivalent manner

$$f(x|X) = \frac{\binom{n}{x}\binom{N-n}{X-x}}{\binom{N}{X}} = \frac{\binom{n}{x}\binom{N-n}{y}}{\binom{N}{x+y}} \tag{3-2}$$

Joint Distribution of x and X

The joint distribution of x and X is

$$f(x, X) = h_N(X)\ f(x|X) \tag{3-3}$$

where $h_N(X)$ is the probability distribution of X, the number of defectives in the lot. This could be considered as the distribution of x and y, since $X = x + y$. Equation (3-3) could be written

$$f(x, y) = h_N(x+y)\ \frac{\binom{n}{x}\binom{N-n}{y}}{\binom{N}{x+y}} \tag{3-4}$$

Distribution of the Number of Defectives in the Sample

The marginal distribution of x is obtained by summing f(x,X) over all possible values of X

$$g_n(x) = \sum_{X=0}^{N} f(x,X) = \binom{n}{x} \sum_{y=0}^{N-n} h_N(x+y) \frac{\binom{N-y}{y}}{\binom{N}{x+y}}, \quad x = 0,1,2,\ldots..n \qquad (3\text{-}5)$$

Hald calls $g_n(x)$ the compound hypergeometric distribution.

Moment of (x, y)

The mean and variance of X may be written in the following form

$$E(X) = N\,\overline{p} \qquad (3\text{-}6)$$

$$V(X) = N\,\overline{p}\,\overline{q}\,(1+\delta_N), \delta_N > -1 \qquad (3\text{-}7)$$

Equation (3-6) defines $\overline{p}$, which can be interpreted as the process average fraction defective. In equation (3-7), $\overline{q} = 1-\overline{p}$ and δ_N is a constant which allows comparison of the variance of the prior distribution with that of a binomial distribution having parameters N and $\overline{p}$. The variance of the prior distribution is said to be subnormal if $\delta_N < 0$, normal if $\delta_N = 0$, and hypernormal if $\delta_N > 0$.

The covariance of x and y is

$$\text{COV}\,(x,y) = \frac{n(N-n)}{N(N-1)}\,[V(X) - N\,\overline{p}\,\overline{q}] \qquad (3\text{-}8)$$

using equation (3-7), the covariance may be written as

$$\text{COV}\ (x,y) = \bar{n}\ \bar{p}\ \bar{q}\ \frac{N-n}{N-1}\ \delta_N \tag{3-9}$$

This result was incorporated into an important theorem by Mood [18].

> The correlation between the number of defective items in the sample and the number of defectives in the remainder of the lot is positive, zero, or negative according as the variance X is greater than, or equal to, or less than the variance, $N\overline{pq}$, of a binomial prior distribution.

This theorem is important because it implies that a decision rule which calls for rejection when x is large is not appropriate for subnormal prior distributions. For such distributions it would be more reasonable to reject when x is small.

Hald shows that the variance of x can be written

$$V(x) = n\ \bar{p}\ \bar{q}\ (1 + \frac{n-1}{N-1}\ \delta_N) \tag{3-10}$$

which means that $g_n(x)$ is hypernormal or subnormal as $h_N(X)$ is hypernormal or subnormal, respectively. This is also true for the marginal distribution of y.

<u>Conditional Distribution of the Number of Defectives in the Uninspected Portion of the Lot, Given the Number of Defectives in the Sample</u>

The conditional distribution of y for a given x is

$$f(y|x) = \frac{f(x,y)}{g_n(x)} \tag{3-11}$$

Hald shows that the mean of this distribution is

$$E(y|x) = (N-n) \frac{(x-1)\ g_{n+1}(x+1)}{(n+1)\ g_n(x)} \tag{3-12}$$

$E(y|x)$ is the expected number of defectives in the uninspected portion of the lot and can be determined when x is observed and the prior distribution is known.

This last result, equation (3-12), is of utmost importance in the economic design of sampling inspection procedures. It is the basis for determining expected losses associated with sentencing lots.

Modified Decision Model

The sequential sampling procedure proposed by Gilbreath is now to be modified. The meaning of symbols used in this discussion is shown in Appendix II, Glossary of Symbols. At any point in the inspection process the following alternatives are considered:

(1) accept now,

(2) reject now,

(3) inspect one more item and reject,

(4) inspect one more item and accept,

(5) inspect two more items and reject, and

(6) inspect two more items and accept.

The decision rule requires termination of inspection at any point, for which the expected decision loss is less than the cost of inspecting one more item plus the expected decision loss if the lot is sentenced at that point, and also less than the cost of inspecting two more items plus the expected decision loss if the lot is sentenced at that point. Otherwise one additional item is inspected and the procedure continues until it is economical to sentence the lot with no further inspection.

The Hypernormal Case

For the hypernormal case the decisions to be made are:

(1) accept the lot now,

(2) reject the lot now,

(3) inspect one more item and reject the lot if the item inspected is bad,

(4) inspect one more item and accept the lot if the item inspected is good,

(5) inspect two more items and reject the lot if the items inspected are bad or if one of them is bad, and

(6) inspect two more items and accept the lot if the two items inspected are good.

Initially, the decision is made to inspect the first item. The expected costs and the decision losses associated with the decision rules are calculated as follows:

The expected loss if the lot is accepted without inspection

$$(K_a = 1)\ E(X) = E(X) \tag{3-13}$$

The expected loss if the lot is rejected without inspection

$$K_r[N-E(X)] \tag{3-14}$$

The costs and the decision losses if the lot is sentenced after inspecting one item is the fixed cost of inspection plus the cost of inspecting one item plus the expected loss if one item is inspected prior to sentencing the lot. These costs and decision losses are

$$K_f + K_s + K_r\ E(y|x = 0)\ \text{(Probability that item is good)}$$
$$+\ K_r[(N-1) - E(y|x = 1)]\ \text{(Probability that item is bad)}$$

or

$$K_f + K_s + K_r\ E(y|x = 0)(1 - E(X)/N) \qquad (3\text{-}15)$$
$$+\ K_r[(N-1) - E(y|x = 1)]\ E(X)/N$$

because

$$\text{Probability that item is good} = \frac{\binom{N-E(X)}{1}\binom{E(X)}{0}}{\binom{N}{1}} = \frac{N-E(X)}{N}$$

$$\text{Probability that item is bad} = \frac{\binom{N-E(X)}{1}\binom{E(X)}{0}}{\binom{N}{1}} = \frac{E(X)}{N}$$

The costs and the decision losses if the lot is sentenced after inspecting two items is the fixed cost of inspection plus the cost of inspecting two items plus the expected loss if two items are inspected prior to sentencing the lot. These costs and decision losses are

$K_f + 2K_s + (K_a = 1)\ E(y|x=0)$ (Probability that both the items are good) $+ K_r[(N-1) - E(y|x=1)]$ (Probability that one of the two items is good and the other bad) $+ K_r[(N-2) - E(y|x=2)]$ (Probability that both the items are bad) or

$$K_f + 2K_s + E(y|x=0)(1-E(X)/N)(1-E(X)/(N-1)$$
$$+ K_r[(N-1) - E(y|x=1)]\ 2(1-E(X)/N)(E(X)/(N-1)$$
$$+ K_r[(N-2) - E(y|x=2)]\ [E(X)/(N-1)][\{E(X)-1\}/N] \quad (3\text{-}16)$$

because

$$(\text{Probability that both the items are good}) = \frac{\binom{N-E(X)}{2}\binom{E(X)}{0}}{\binom{N}{2}}$$

$$= (1-E(X)/N)(1-E(X)/(N-1))$$

(Probability that one of the itmes is good and other is bad)

$$= \frac{\binom{N-E(X)}{1}\binom{X}{1}}{\binom{N}{2}} = 2(1-E(X)/N)(E(X)/(N-1))$$

$$(\text{Probability that both the items are bad}) = \frac{\binom{N-E(X)}{0}\binom{E(X)}{2}}{\binom{N}{2}}$$

$$= \frac{E(X)(E(X)-1)}{N(N-1)}$$

The decision is then made, if

$$\text{Min}(3\text{-}13,\ 3\text{-}14) \leq (3\text{-}15) \text{ and } \leq (3\text{-}16)$$

inspect nothing and either accept or reject according to whether (3-13) or (3-14) is less. If

$$\text{Min}(3\text{-}13),\ 3\text{-}14) \leq (3\text{-}15) \text{ and } > (3\text{-}16)$$

or

$$\text{Min}(3\text{-}13,\ 3\text{-}14) > (3\text{-}15) \text{ and } \leq (3\text{-}16)$$

or

$$\text{Min}(3\text{-}13,\ 3\text{-}15) > (3\text{-}15) \text{ and } > (3\text{-}16)$$

inspect one item.

Iterative Decision Procedure

When the decision is made to inspect the first item, the procedure becomes general. If at any point in the inspection procedure n items have been inspected and x defectives have been found, the decision losses are

Accept now

$$(K_a = 1)\ E(y|x_n) \tag{3-17}$$

Reject now

$$K_r[(N-n) - E(y|x_n)] \tag{3-18}$$

Inspect one more item and sentence

$$K_s + (K_a = 1)\ E(y|x_{n+1})\left[1 - \frac{E(y|x_n)}{N-n}\right] + K_r\left[(N-n-1) - E(y|x+1_{n+1})\right] E(y|x_n)/(N-n) \tag{3-19}$$

Inspect two more items and sentence

$$2K_s + (K_a = 1)\ E(y|x_{n+2})\left[1 - \frac{E(y|x_{n+2})}{N-n}\right]\left[1 - \frac{E(y|x_{n+2})}{N-n-1}\right] + K_r\left[(N-n-1) - E(y|x+1_{n+2})\right]2\left[1 - \frac{E(y|x+1_{n+2})}{N-n}\right]\left[\frac{E(y|x+1_{n+2})}{N-n-1}\right] \tag{3-20}$$

$$+ K_r \left[(N-n-2) - E(y|x+2_{n+2})\right]\left[\frac{E(y|x+2_{n+2})\ (E(y|x+2_{n+2}) - 1)}{(N-n)(N-n-1)}\right]$$

The decision is then made, if

$$\text{Min}(3\text{-}17,\ 3\text{-}18) \leqq (3\text{-}19) \text{ and } \leqq (3\text{-}20)$$

inspect nothing and terminate inspection. Select the appropriate decision corresponding to Min(3-17, 3-18). If

$$\text{Min}(3\text{-}17,\ 3\text{-}18) \leqq (3\text{-}19) \text{ and } > (3\text{-}20)$$

or

$$\text{Min}(3\text{-}17,\ 3\text{-}18) > (3\text{-}19) \text{ and } \leqq (3\text{-}20)$$

or

$$\text{Min}(3\text{-}17,\ 3\text{-}18) > (3\text{-}19) \text{ and } > (3\text{-}20)$$

inspect one more item, set (n+1) equal to n and repeat the procedure until it is economical to sentence the lot without inspecting another item.

The Subnormal Case

x and y are negatively correlated and discovery of a defective item would suggest acceptance. The alternatives for a decision at any point in the inspection are:

(1) accept now,

(2) reject now,

(3) inspect one more item and reject the lot if the item inspected is good,

(4) inspect one more item and accept the lot if the item inspected is bad,

(5) inspect two more items and reject the lot if the items inspected are good, or if one of them is good, and

(6) inspect two more items and accept the lot if the two items inspected are bad.

Initially, the decision is made to inspect the first item. The expected costs and the decision losses associated with the decision rules are calculated as follows:

Accept now

$$K_a E(X) = E(X) \tag{3-21}$$

Reject now

$$K_r[N-E(X)] \tag{3-22}$$

Inspect one item and sentence the lot

$$K_f + K_s + E(y|x = 0)\ E(X)/N + K_r[(N-1) - E(y|x = 1)][1 - E(X)/N] \tag{3-23}$$

Inspect two items and sentence the lot

$$\begin{aligned} &K_f + 2K_s + E(y|x = 0)[E(X)/(N-1)][(E(X) - 1)/N] \\ &+ [(N-1) - E(y|x = 1)]2[1 - E(X)/N]E(X)/(N-1) \\ &+ k_r[(N-2) - E(y|x = 2)][(1 - E(X)/N)][1 - E(X)/(N-1)] \end{aligned} \tag{3-24}$$

If Min(3-21, 3-22) ≤ (3-23) and ≤ (3-24) inspect nothing and sentence the lot without inspection, accepting it or rejecting it depending on Min(3-21, 3-22). Otherwise, inspect one item.

Iteration Decision

When the decision is made to inspect the first item, the procedure becomes general. If at any point during inspection n items have been inspected and x defectives have been found, the decision losses are

Accept now

$$(K_a = 1)\ E(y|x_n) \tag{3-25}$$

Reject now

$$K_r[(N-1) - E(y|x_n)] \tag{3-26}$$

Inspect one more item and sentence the lot

$$K_s + (K_a = 1)\ E(y|x_{n+1})[E(y|x_n)/(N-n)] + K_r[(N-n-1) - E(y|x+1_{n+1})][1 - E(y|x_n)/(N-n)] \tag{3-27}$$

Inspect two more items and sentence the lot

$$2K_s + E(y|x_{n+2})[E(y|x_{n+2})/(N-n)](E(y|x_{n+2}) - 1)/(N-n-1) + [(N-n-1) - E(y|x+1_{n+1})]2[1 - E(y|x+1_{n+2})/(N-n)] \cdot [E(y|x+1_{n+2})/(N-n-1)] + K_r[(N-n-2) - E(y|x+2_{n+2})] \cdot [1 - E(y|x+2_{n+2})/(N-n)][1 - E(y|x+2_{n+2})/(N-n-1)] \tag{3-28}$$

If Min(3-25, 3-26) $\leq$ (3-27) and $\leq$ (3-28) terminate inspection and sentence the lot, otherwise inspect one more item, set (n+1) equal to n and repeat the procedure until it is economical to sentence the lot without inspecting another item.

The Normal Case

In the case of a normal prior distribution, x and y are uncorrelated, and no information about the remainder of the lot can be gained by sampling inspection. In this case, acceptance or rejection without inspection are the only logical alternatives [8].

The cost of inspecting all previous items is a sunk cost with regard to future or present decisions. Therefore, it is logical to consider the cost of inspecting the first item, the total fixed cost of sampling inspection plus the average variable cost of inspecting one item. Beyond the first item, only the average variable cost of sampling inspection is relevant to the decision process [8].

CHAPTER IV

EVALUATION OF THE MODIFIED DECISION MODEL

The modified sequential sampling procedure is tested for a two-component mixed binomial prior distribution. The development of the component mixed binomial prior distribution appears in Gilbreath's work [8] as follows.

If a process generates lots in which the probability of a defective is constant during the production of a lot but varies from lot to lot according to a given weight function, $w(p)$, the distribution of X for M possible levels is

$$h_N(X;\ p_i, w_i) = \sum_{i=1}^{M} w_i \binom{N}{X} p_i^{X}\ q_i^{N-X},\quad X = 0, 1, 2, \ldots\ldots, N \qquad (4\text{-}1)$$

where

$$\sum_{i=1}^{M} w_i = 1, \quad \text{and } w_i \geq 0, \qquad i = 1, 2, 3, \ldots\ldots\ldots, M.$$

The mean is

$$E(X) = N\,\overline{p}$$

where

$$\overline{p} = \sum_{i=1}^{M} w_i\ p_i$$

The variance is

$$V(X) = N \sum_{i=1}^{M} w_i\ p_i\ q_i + N^2 \sum_{i=1}^{M} w_i (p_i - \overline{p})^2 \qquad (4\text{-}2)$$

The mixed binomial distribution is hypernormal, since

$$\delta_N = (N-1) \sum_{i=1}^{M} \frac{w_i(p_i - \bar{p})^2}{\bar{p}\,\bar{q}} \tag{4-3}$$

which is positive for $N > 1$.

The expected number of defective items in the uninspected portion of lots from which x defectives have been found in a sample of n items is

$$E(y|x) = (N-n) \sum_{i=1}^{M} w_i(x)\; p_i \tag{4-4}$$

where

$$w_i(x) = \frac{w_i\, p_i^{\,x}\, q_i^{\,n-x}}{\sum_{i=1}^{M} w_i\, p_i^{\,x}\, q_i^{\,n-x}} \tag{4-5}$$

Equation (4-4) may be written in expanded form using (4-5) as

$$E(y|x) = (N-n) \sum_{i=1}^{M} \frac{w_i\, p_i^{\,x}\, q_i^{\,n-x}}{w_i\, p_i^{\,x}\, q_i^{\,n-x}}\, p_i \tag{4-6}$$

Decision Model for Mixed Binomial Prior Distribution

Initial Decision

First, the decision is made whether to inspect the first item. From (3-13) and (3-14) choose

$$\text{Min}\left[N\sum_{i=1}^{M} w_i\ p_i,\ K_r(N-N\sum_{i=1}^{M} w_i\ p_i)\right] \tag{4-7}$$

which is the minimum loss if the lot is sentenced without inspection. Compare this loss to the loss if the lot is sentenced after inspecting one item and to the loss if the lot is sentenced after inspecting two items. From (3-15)

$$K_f + K_s + (N-1)\sum_{i=1}^{M}\left(\frac{w_i\ q_i}{\sum_{i=1}^{M} w_i\ q_i}\ p_i\right)(1 - w_i\ p_i) \tag{4-8}$$

$$+ K_r(N-1)\left(1 - \sum_{i=1}^{M}\frac{w_i\ p_i}{\sum_{i=1}^{M} w_i\ p_i}\ p_i\right)\sum_{i=1}^{M} w_i\ p_i$$

From (3-16)

$$K_f + 2K_s + (N-2)\left[\sum_{i=1}^{M}\frac{w_i\ q_i^2}{\sum_{i=1}^{M} w_i\ q_i^2}\ p_i\right]\left[1 - \sum_{i=1}^{M} w_i\ p_i\right] \tag{4-9}$$

$$\cdot\left[1 - \frac{N}{N-1}\sum_{i=1}^{M} w_i\ p_i\right] + K_r\left[(N-1) - (N-2)\sum_{i=1}^{M}\frac{w_i\ p_i\ q_i}{\sum_{i=1}^{M} w_i\ p_i\ q_i}\ p_i\right]$$

$$\cdot\ 2\left(1 - \sum_{i=1}^{M} w_i\ p_i\right)\left(\frac{N}{N-1}\sum_{i=1}^{M} w_i\ p_i\right) + \left[K_r\ (N-2) - (N-2)\sum_{i=1}^{M}\frac{w_i\ p_i^2}{\sum_{i=1}^{M} w_i\ p_i^2}\ p_i\right]$$

$$\cdot \left[\frac{N}{N-1} \sum_{i=1}^{M} w_i \; p_i\right]\left(\sum_{i=1}^{M} w_i \; p_i \; - 1/N\right)$$

If (4-7) ≦ (4-8) and ≦ (4-9) inspect nothing and sentence the lot. Otherwise inspect one item.

Iterative Decision Procedure

Once the decision is made to inspect one item, the decision procedure is general. From (3-17) and (3-18), choose

$$\text{Min}\left[(N-n) \sum_{i=1}^{M} \frac{w_i \; p_i^{\,x} \; q_i^{\,n-x}}{\sum_{i=1}^{M} w_i \; p_i^{\,x} \; q_i^{\,n-x}} \; p_i, \right. \tag{4-10}$$

$$\left. K_r(N-n)\left(1 - \sum_{i=1}^{M} \frac{w_i \; p_i^{\,x} \; q_i^{\,n-x}}{\sum_{i=1}^{M} w_i \; p_i^{\,x} \; q_i^{\,n-x}} \; p_i\right)\right]$$

Compare this loss to the loss if the lot is sentenced after inspecting (n+1) items and to the loss if the lot is sentenced after inspecting (n+2) items. From (3-19)

$$K_s + (N-n-1) \sum_{i=1}^{M} \frac{w_i^{\,x} \; p_i \; q_i^{\,n-x+1}}{\sum_{i=1}^{M} w_i \; p_i^{\,x} \; q_i^{\,n-x+1}} \; p_i \tag{4-11}$$

$$\cdot \left[1 - \sum_{i=1}^{M} \frac{w_i \; p_i^{\,x} \; q_i^{\,n-x}}{\sum_{i=1}^{M} w_i \; p_i^{\,x} \; q_i^{\,n-x}} \; p_i\right] + K_r(N-n-1)$$

$$\cdot \left[1 - \sum_{i=1}^{M} \frac{w_i\, p_i^{x+1}\, q_i^{n-x}}{\sum_{i=1}^{M} w_i\, p_i^{x+1}\, q_i^{n-x}}\, p_i\right]\left[\sum_{i=1}^{M} \frac{w_i\, p_i^{x}\, q_i^{n-x}}{\sum_{i=1}^{M} w_i\, p_i^{x}\, q_i^{n-x}}\, p_i\right]$$

From (3-20)

$$2K_s + [(N-n-2)]\left[\sum_{i=1}^{M} \frac{w_i\, p_i^{x}\, q_i^{n-x+2}}{\sum_{i=1}^{M} w_i\, p_i^{x}\, q_i^{n-x+2}}\, p_i\right] \tag{4-12}$$

$$\cdot \left[1 - \frac{N-n-2}{N-n} \sum_{i=1}^{M} \frac{w_i\, p_i^{x}\, q_i^{n-x+2}}{\sum_{i=1}^{M} w_i\, p_i^{x}\, q_i^{n-x+2}}\, p_i\right]$$

$$\cdot \left[1 - \frac{N-n-2}{N-n-1} \sum_{i=1}^{M} \frac{w_i\, p_i^{x}\, q_i^{n-x+2}}{\sum_{i=1}^{M} w_i\, p_i^{x}\, q_i^{n-x+2}}\, p_i\right]$$

$$+ K_r\left[(N-n-1) - (N-n-2) \sum_{i=1}^{M} \frac{w_i\, p_i^{x+1}\, q_i^{n-x+1}}{\sum_{i=1}^{M} w_i\, p_i^{x+1}\, q_i^{n-x+1}}\, p_i\right]$$

$$\cdot\, 2\left[1 - \frac{N-n-2}{N-n} \sum_{i=1}^{M} \frac{w_i\, p_i^{x+1}\, q_i^{n-x+1}}{\sum_{i=1}^{M} w_i\, p_i^{x+1}\, q_i^{n-x+1}}\, p_i\right]$$

$$\cdot \left[\frac{N-n-2}{N-n-1} \sum_{i=1}^{M} \frac{w_i\, p_i^{x+1}\, q_i^{n-x+1}}{\sum_{i=1}^{M} w_i\, p_i^{x+1}\, q_i^{n-x+1}}\, p_i\right]$$

$$+ K_r \left[(N-n-2) - (N-n-2) \sum_{i=1}^{M} \frac{w_i \, p_i^{x+2} \, q_i^{n-x}}{\sum_{i=1}^{M} w_i \, p_i^{x+2} \, q_i^{n-x}} \, p_i \right]$$

$$\cdot \left[\frac{N-n-2}{(N-n)(N-n-1)} \sum_{i=1}^{M} \frac{w_i \, p_i^{x+2} \, q_i^{n-x}}{\sum_{i=1}^{M} w_i \, p_i^{x+2} \, q_i^{n-x}} \, p_i \right]$$

$$\cdot \left[(N-n-2) \sum_{i=1}^{M} \frac{w_i \, p_i \, q_i}{\sum_{i=1}^{M} w_i \, p_i^{x-2} \, q_i^{n-2}} \, p_i - 1 \right]$$

If (4-10) $\leq$ (4-11) and $\leq$ (4-12) then terminate inspection after inspecting n items and sentence the lot. Otherwise set (n+1) equal to n and repeat the iterative procedure until it is economical to sentence the lot without further inspection.

Programming of the Modified Decision Model

Gilbreath programmed his decision model for the Burroughs 220 electronic digital computer. His program was appropriately altered for the Burroughs B-5500 computer with necessary changes for the modified decision model. The modified sequential procedure was then tested for the two-component mixed binomial prior distribution. The program was written for an M component mixed binomial prior distribution and works for M $\leq$ 12. The program could even be used for more than twelve components with some minor changes.

Simulation of Lots

Gilbreath used a simulation procedure in which lots of size N are generated as arrays, X(0), each time a previous lot is sentenced. For each N at all points in the test procedure, 100 lots are generated and sentenced. The array is then filled with zeros and ones, zeros representing good items and ones representing bad items. The ones are added for each lot generated. The sum is the number of defectives in the generated lot. Twelve-digit random numbers are used to determine the proportion fraction defective p_1 of the lot and to determine the weight factor w_i of each fraction defective p_i. Figure 1 [8] illustrates the lot generation process.

Sampling Procedure

The sampling procedure used by Gilbreath uses ten-digit random numbers for selecting each item in a sample from the lot stored in the computer memory. In this case, twelve-digit random numbers are used. A standard procedure stored in the computer memory is utilized to generate twelve-digit random numbers. The random numbers are generated such that these lie between 0 and 1. To achieve integer values, the ENTIER instruction for the Burroughs B-5500 is employed. Figure 2 [8] illustrates the sampling procedure.

Cost Analysis

Values of lot sizes N_1, N_2, N_3, ..., process fraction defectives p_1 and p_2, and weight factors w_1 and w_2 are fed into the computer for each combination of cost parameters K_s and K_r. K_f is always held zero. The values of the various parameters used are kept the same as in Hald's and Gilbreath's work. This is to compare the efficiency of this procedure to Hald's optimum single sampling procedure and to Gilbreath's sequential sampling procedure. One hundred lots are generated and sentenced for each

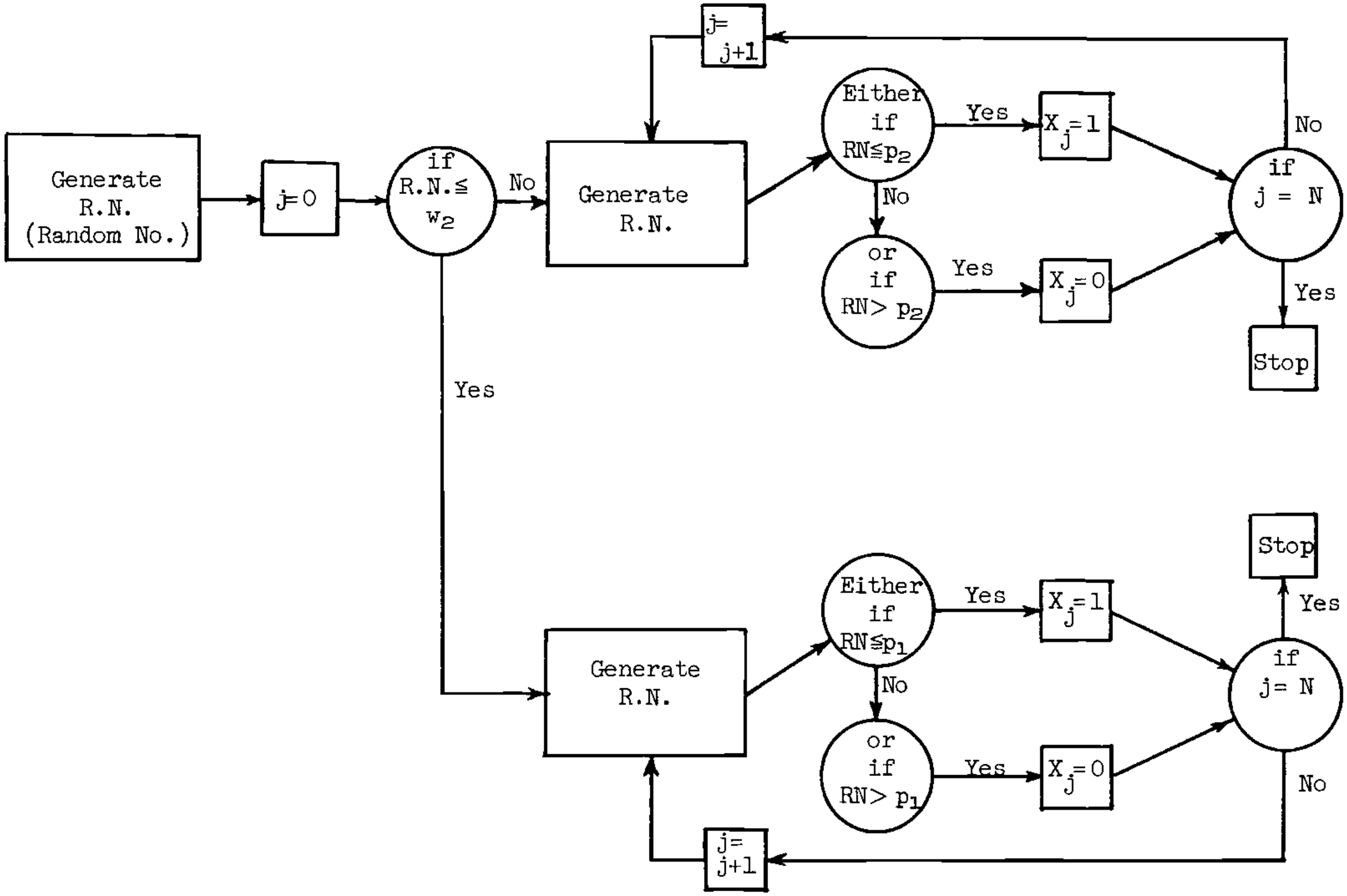

Figure 1. Lot Generation Procedure

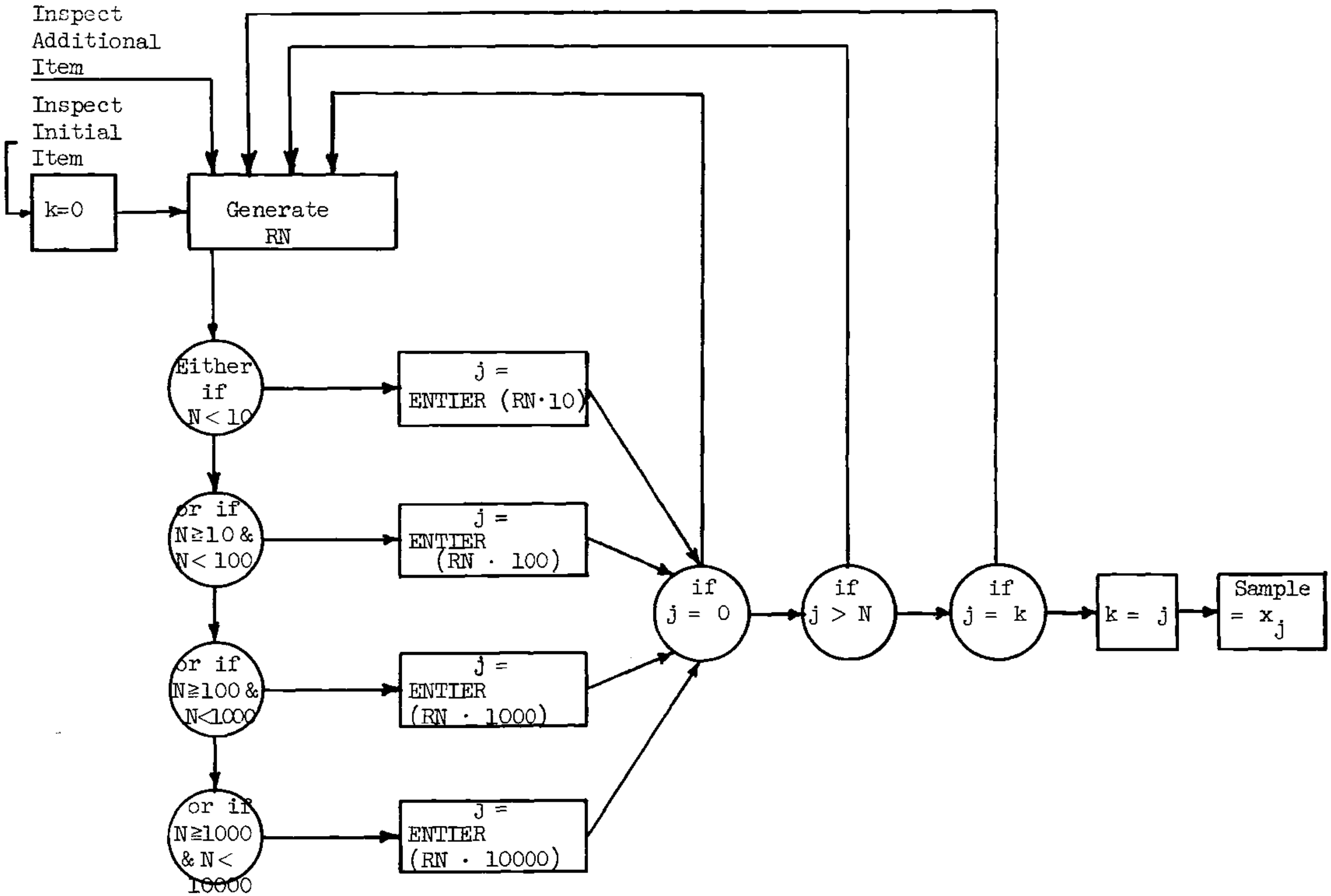

Figure 2. Sampling Procedure

lot size. The number of defectives in the generated lot, sample size, number of defectives in the sample, decision whether to accept or reject the lot after the lot is sentenced, and the total of inspection costs and decision losses are computed and recorded for each lot generated. The totals of the inspection costs and the decision losses computed for each lot generated are added and recorded. The sample sizes and the costs computed are averaged over the 100 lots to arrive at the average sample size and average unit cost. The summary of these calculations along with the results of Hald's optimum single sampling procedure and Gilbreath's sequential sampling procedure appear in Tables 1, 2, 3, 4, and 5. Unit costs of the modified sequential procedure are compared with the unit costs of sentencing the lot without inspection and of the use of Hald's and Gilbreath's procedures.

<u>Model Effectiveness</u>

The proposed sequential procedure is evaluated for three values of $(p_2 - p_1)$

(1) $p_2 = 0.5$, $p_1 = 0.1$, $(p_2 - p_1) = 0.4$

$K_r = K_s = 0.2$ $\quad K_f = 0.0$

(2) $p_2 = 0.3$, $p_1 = 0.06$, $(p_2 - p_1) = 0.24$

$K_r = K_s = 0.12$, and

(3) $p_1 = 0.1$, $p_2 = 0.02$, $(p_2 - p_1) = 0.08$

$K_r = K_s = 0.05$

<u>When $p_2 - p_1$ Is Small ($p_2 = 0.1$ and $p_1 = 0.02$)</u>. The results are most favorable for this case. The test for the differences in unit costs is applied to the single sampling versus the modified sequential procedure

Table 1. Data for the Modified Sequential Sampling Procedure, Gilbreath's Sequential Sampling Plan, and Hald's Optimum Single Sampling Plan
$p_1 = 0.1$, $p_2 = 0.5$, $K_s = K_r = 0.2$, $w_1 = 0.8$, $w_2 = 0.2$

Average Lot Size N	Hald's Optimum Single Sample Size n	Gilbreath's Sequential Average Sample Size $\bar{n}$	Modified Sequential Average Sample Size $\bar{\bar{n}}$	Cost of Accepting Without Inspection $\$/C_a$	Expected Cost Hald's Optimum $\$/C_a$	Average Cost Gilbreath's Sequential $\$/C_a$	Average Cost Modified Sequential $\$/C_a$
4	1	1	1	0.1800	0.1598	0.1450	0.1900*
12	2	1.82	1.91	0.1800	0.1435	0.1448	0.1313
25	5	2.38	2.77	0.1800	0.1362	0.1434	0.1246
38	6	2.48	2.55	0.1800	0.1294	0.1375	0.1323
41	8	2.57	2.53	0.1800	0.1307	0.1365	0.1296
57	9	2.57	2.62	0.1800	0.1247	0.1395	0.1200
85	12	2.56	2.51	0.1800	0.1205	0.1280	0.1270
113	13	2.49	2.54	0.1800	0.1167	0.1192	0.1261
165	16	3.52	3.57	0.1800	0.1135	0.1236	0.1266
213	17	3.39	3.75	0.1800	0.1111	0.1209	0.1287
297	20	3.29	3.58	0.1800	0.1090	0.1243	0.1208
492	24	3.37	3.32	0.1800	0.1062	0.1262	0.1247
680	27	3.77	3.45	0.1800	0.1049	0.1271	0.1227
859	28	3.37	3.53	0.1800	0.1041	0.1326	0.1227

*This value is not considered in the analysis because this value deviates too much from other values.

Table 2. Data for the Modified Sequential Sampling Procedure, Gilbreath's Sequential Sampling Plan, and Hald's Optimum Single Sampling Plan
$p_1 = 0.1$, $p_2 = 0.5$, $K_s = 0.2$, $K_r = 0.3$, $w_1 = 0.8$, $w_2 = 0.2$

Average Lot Size N	Hald's Optimum Single Sample Size n	Gilbreath's Sequential Average Sample Size $\bar{n}$	Modified Sequential Average Sample Size $\bar{n}$	Cost of Accepting Without Inspection $\$/C_a$	Expected Cost Hald's Optimum $\$/C_a$	Average Cost Gilbreath's Sequential $\$/C_a$	Average Cost Modified Sequential $\$/C_a$
5	0	1	1	0.18000	0.18000	0.17480	0.18680
33	3	2.09	2.38	0.18000	0.15345	0.15745	0.14391
51	4	2.11	2.12	0.18000	0.14378	0.14010	0.15133
84	7	2.23	2.19	0.18000	0.13524	0.14640	0.14213
115	10	2.40	2.00	0.18000	0.13040	0.14591	0.16561
154	11	2.23	2.07	0.18000	0.12614	0.13021	0.15890
224	14	2.24	2.26	0.18000	0.12269	0.16221	0.15344
291	15	2.21	2.35	0.18000	0.12020	0.12303	0.13672
426	18	2.02	2.32	0.18000	0.11786	0.15929	0.14203
541	19	2.30	2.28	0.18000	0.11642	0.15920	0.16127
715	22	2.25	2.24	0.18000	0.11512	0.14279	0.14728
942	25	2.38	2.37	0.18000	0.11416	0.14359	0.13666

Table 3. Data for the Modified Sequential Sampling Procedure, Gilbreath's Sequential Sampling Plan, and Hald's Optimum Single Sampling Plan
$p_1 = 0.1$, $p_2 = 0.5$, $K_s = 0.2$, $K_r = 0.4$, $w_1 = 0.8$, $w_2 = 0.2$

Average Lot Size N	Hald's Optimum Single Sample Size n	Gilbreath's Sequential Average Sample Size $\bar{n}$	Modified Sequential Average Sample Size $\bar{n}$	Cost of Accepting Without Inspection $\$/C_a$	Expected Cost Hald's Optimum $\$/C_a$	Average Cost Gilbreath's Sequential $\$/C_a$	Average Cost Modified Sequential $\$/C_a$
12	0	1.44	1.52	0.18000	0.18000	0.16983	0.16583
157	5	1.62	1.23	0.18000	0.15332	0.16032	0.14281
173	8	1.70	1.51	0.18000	0.14549	0.18424	0.17177
243	9	1.36	1.59	0.18000	0.13929	0.17002	0.16284
343	12	1.40	1.55	0.18000	0.13490	0.14942	0.18259
439	13	1.50	1.51	0.18000	0.13144	0.15907	0.17578
629	16	1.41	1.52	0.18000	0.12886	0.13350	0.15634
796	17	1.87	1.47	0.18000	0.12689	0.14932	0.15272

Table 4. Data for the Modified Sequential Sampling Procedure, Gilbreath's Sequential Sampling Plan, and Hald's Optimum Single Sampling Plan
$p_1 = 0.06$, $p_2 = 0.30$, $K_s = 0.12$, $K_r = 0.12$, $w_1 = 0.8$, $w_2 = 0.20$

Average Lot Size N	Hald's Optimum Single Sample Size n	Gilbreath's Sequential Average Sample Size $\bar{n}$	Modified Sequential Average Sample Size $\bar{n}$	Cost of Accepting Without Inspection $\$/C_a$	Expected Cost Hald's Optimum $\$/C_a$	Average Cost Gilbreath's Sequential $\$/C_a$	Average Cost Modified Sequential $\$/C_a$
44	9	4.12	4.16	0.10800	0.08612	0.08985	0.08965
92	15	3.97	4.09	0.10800	0.08040	0.08742	0.09385
162	22	5.23	5.22	0.10800	0.07628	0.07547	0.08065
260	28	4.82	4.83	0.10800	0.07350	0.08912	0.08836
407	34	4.92	5.04	0.10800	0.07131	0.07846	0.07996
642	41	4.91	5.15	0.10800	0.07059	0.08005	0.08547
971	57	4.84	5.18	0.10800	0.06731	0.08106	0.08199

Table 5. Data for the Modified Sequential Sampling Procedure, Gilbreath's Sequential Sampling Plan, and Hald's Optimum Single Sampling Plan
$p_1 = 0.02$, $p_2 = 0.10$, $K_s = 0.05$, $K_r = 0.05$, $w_1 = 0.8$, $w_2 = 0.2$

Average Lot Size N	Hald's Optimum Single Sample Size n	Gilbreath's Sequential Average Sample Size $\bar{n}$	Modified Sequential Average Sample Size $\bar{n}$	Cost of Accepting Without Inspection $\$/C_a$	Expected Cost Hald's Optimum $\$/C_a$	Average Cost Gilbreath's Sequential $\$/C_a$	Average Cost Modified Sequential $\$/C_a$
303	40	7.96	8.28	0.03600	0.03134	0.03500	0.03033
528	60	8.85	8.41	0.03600	0.02984	0.03263	0.03397
850	80	8.64	8.30	0.03600	0.02870	0.03379	0.03607
1000	--	----	8.74	0.03600	----	----	0.03239

and the proposed sequential procedure versus the modified sequential procedure. There are no significant differences among the three procedures. The average economic difference between the modified sequential procedure and the optimum single sampling, and the economic difference between the modified sequential procedure and the single sampling are $0.000139/C_a$ and $0.003503/C_a$, respectively.

When $p_2 - p_1$ Is Intermediate ($p_2 = 0.3$ and $p_1 = 0.06$). The "t" test for the differences in unit costs was applied. There is no significant difference between the unit costs of Gilbreath's sequential procedure and the modified sequential procedure. The average difference is $0.004724/C_a$. Hald's single sampling procedure tested significantly superior to the modified sequential sampling. The average difference is $0.01049/C_a$.

When $p_2 - p_1$ Is Large ($p_2 = 0.05$ and $p_1 = 0.1$). The same test was applied to these results. There is not a significant difference in the unit costs of the modified sequential procedure and Gilbreath's sequential procedure. The average difference is $-0.005052/C_a$. Hald's single sampling procedure is significantly superior to the modified sequential procedure. The average difference in unit costs between the modified sequential procedure and the single sampling plan is $0.007545/C_a$.

Figure 3 shows the comparison of average unit costs for the modified sequential plan, Gilbreath's sequential plan, Hald's single sampling plan, and acceptance of the lot without inspection. Tables 1, 4, and 5 and Figure 3 show the effect of changes in $p_2 - p_1$ on the unit costs of inspection.

Probability of Acceptance

For each lot size, 100 lots were generated and sentenced. The prob-

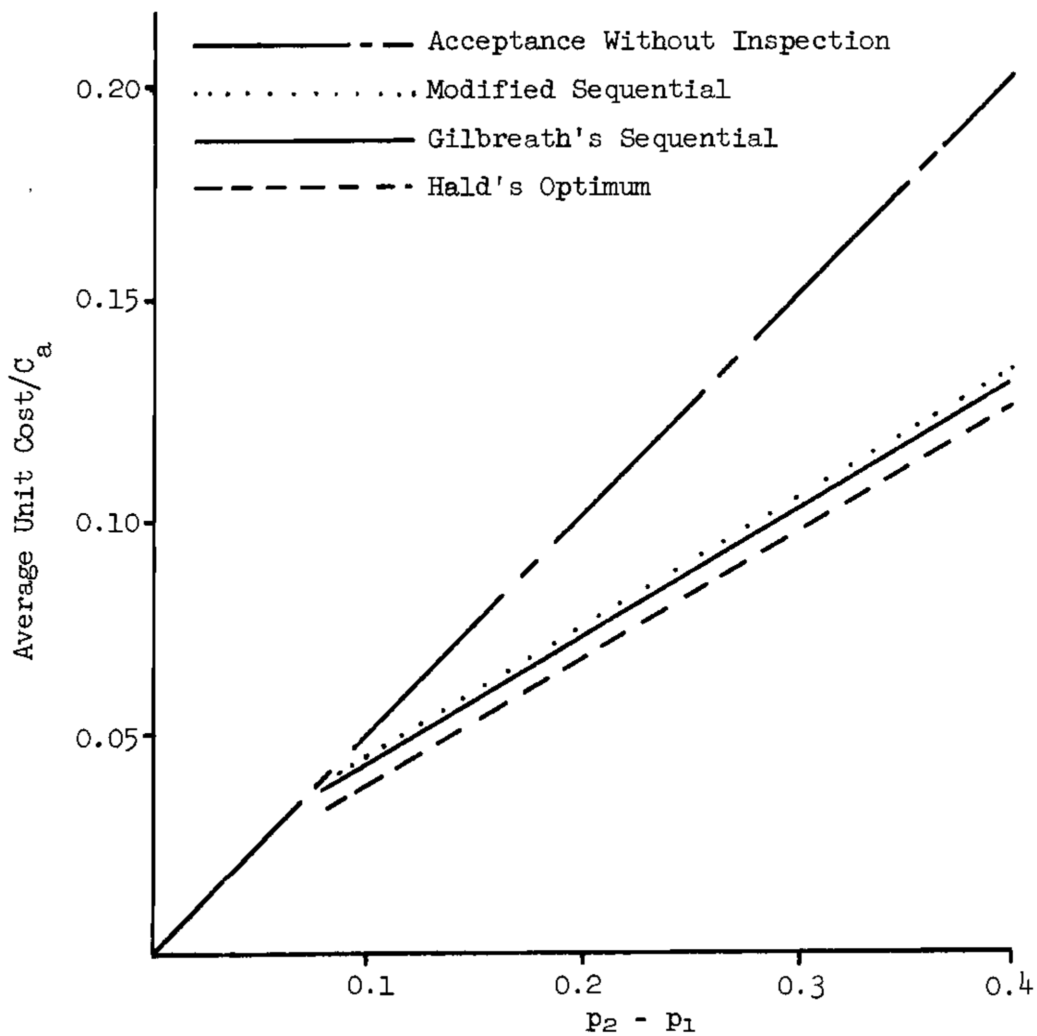

Figure 3. Comparison of Unit Costs for Values of $(p_2 - p_1)$; $p_1 < p_2$

ability of acceptance is calculated as the ratio of the number of generated lots accepted to the number of lots generated and sentenced. Average Outgoing Quality (A.O.Q.) for all values of $p_2 - p_1$ is investigated. Figures 4, 5, 6, 7, and 8 show the comparable probabilities of acceptance for single sampling, Gilbreath's proposed sequential sampling, and the modified sequential sampling procedures. The probability of acceptance for the modified sequential procedure is in all cases comparable to Gilbreath's sequential procedure, but it is less than that corresponding probability for the single sampling procedure. The average sample sizes for the modified sequential plans are less than corresponding single sampling plans but are comparable to Gilbreath's sequential plans. The modified sequential procedure gives no cost advantage over Gilbreath's sequential procedure. The optimum single sampling procedure is more economical than the modified sequential procedure.

Sensitivity Analysis of the Modified Sequential Procedure

Effect of the Change in Cost Parameters

For $p_1 = 0.1$ and $p_2 = 0.5$, K_r is assigned three different values, $K_r = 0.2$, 0.3, and 0.4, and the model is evaluated. The model is compared with Hald's single sampling and Gilbreath's sequential sampling procedures. These values are tabulated in Tables 1, 2, and 3. There is not a significant difference in the unit costs of Gilbreath's sequential procedure and the modified sequential procedure. In the first two cases, the optimum single sampling is more economical than the modified sequential sampling. In the third case, there is no significant difference in the unit costs between the modified sequential sampling and the optimum single sampling procedures.

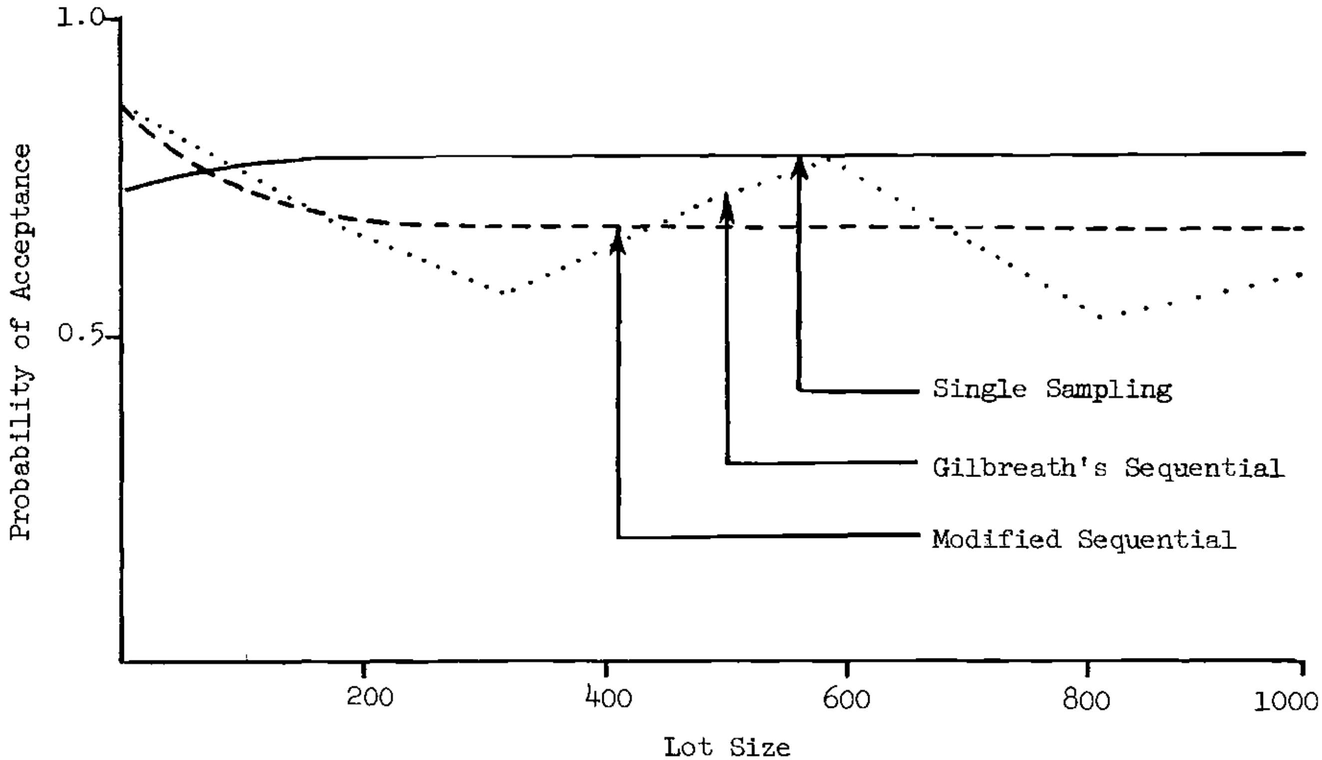

Figure 4. Comparison of Acceptance Probabilities
$p_1 = 0.1$, $p_2 = 0.5$, $K_r = K_s = 0.2$, $K_f = 0.0$

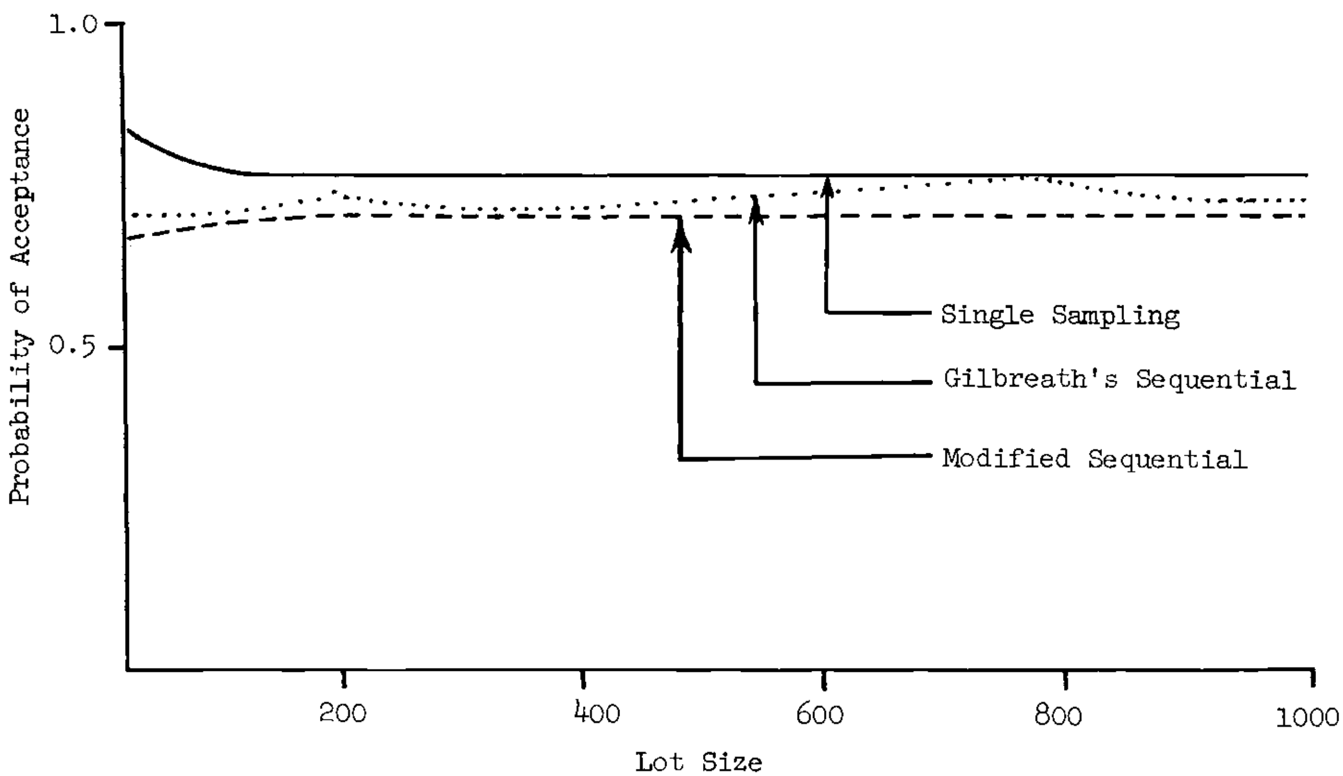

Figure 5. Comparison of Acceptance Probabilities
$p_1 = 0.1$, $p_2 = 0.5$, $K_r = 0.3$, $K_s = 0.2$

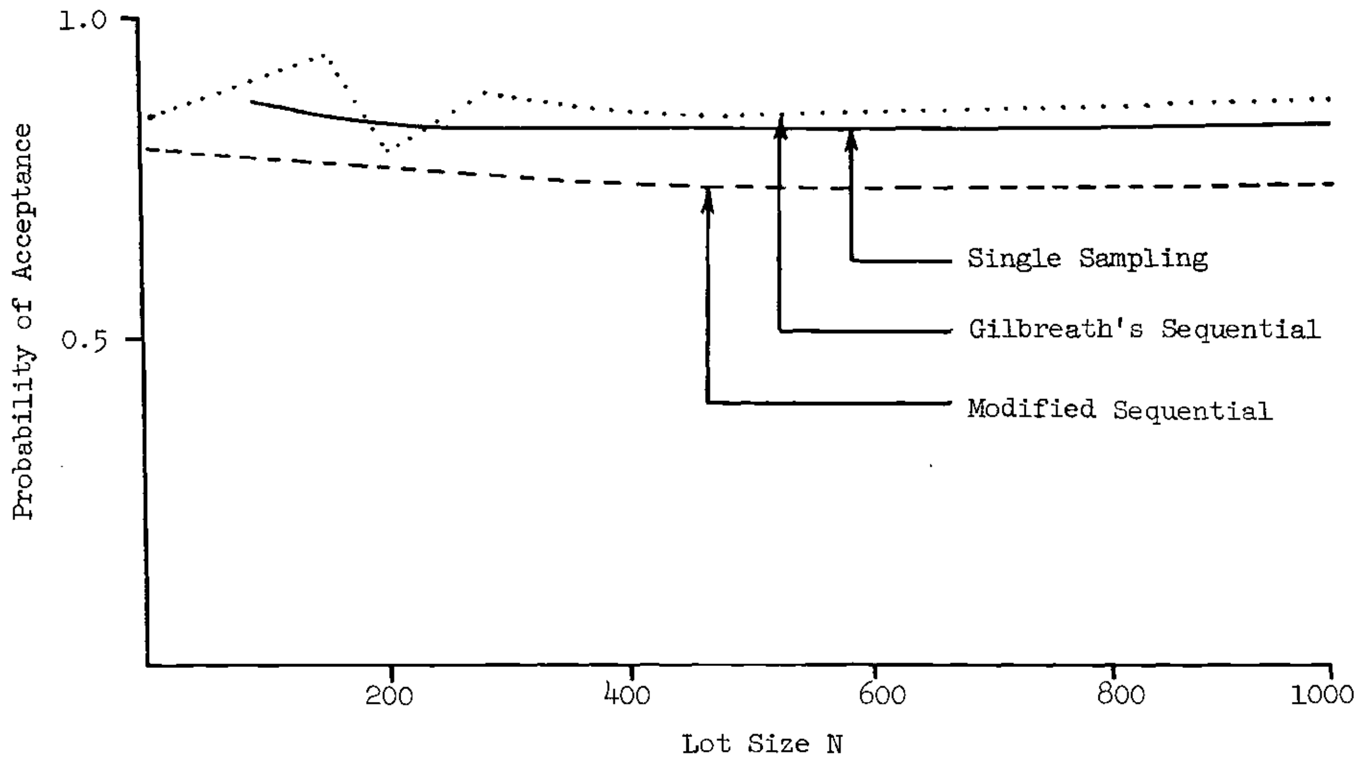

Figure 6. Comparison of Acceptance Probabilities
$p_1 = 0.1$, $p_2 = 0.5$, $K_s = 0.2$, $K_r = 0.4$

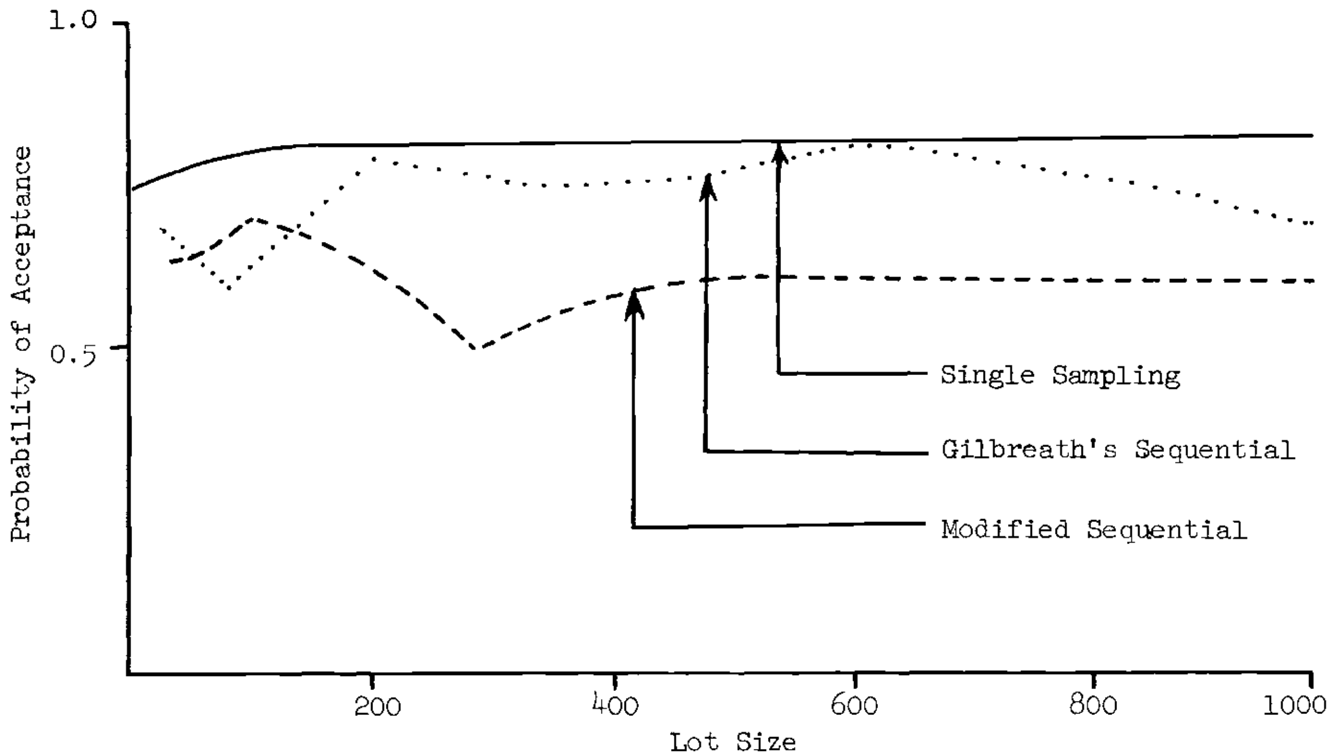

Figure 7. Comparison of Acceptance Probabilities
$p_1 = 0.06$, $p_2 = 0.30$, $K_s = K_r = 0.12$

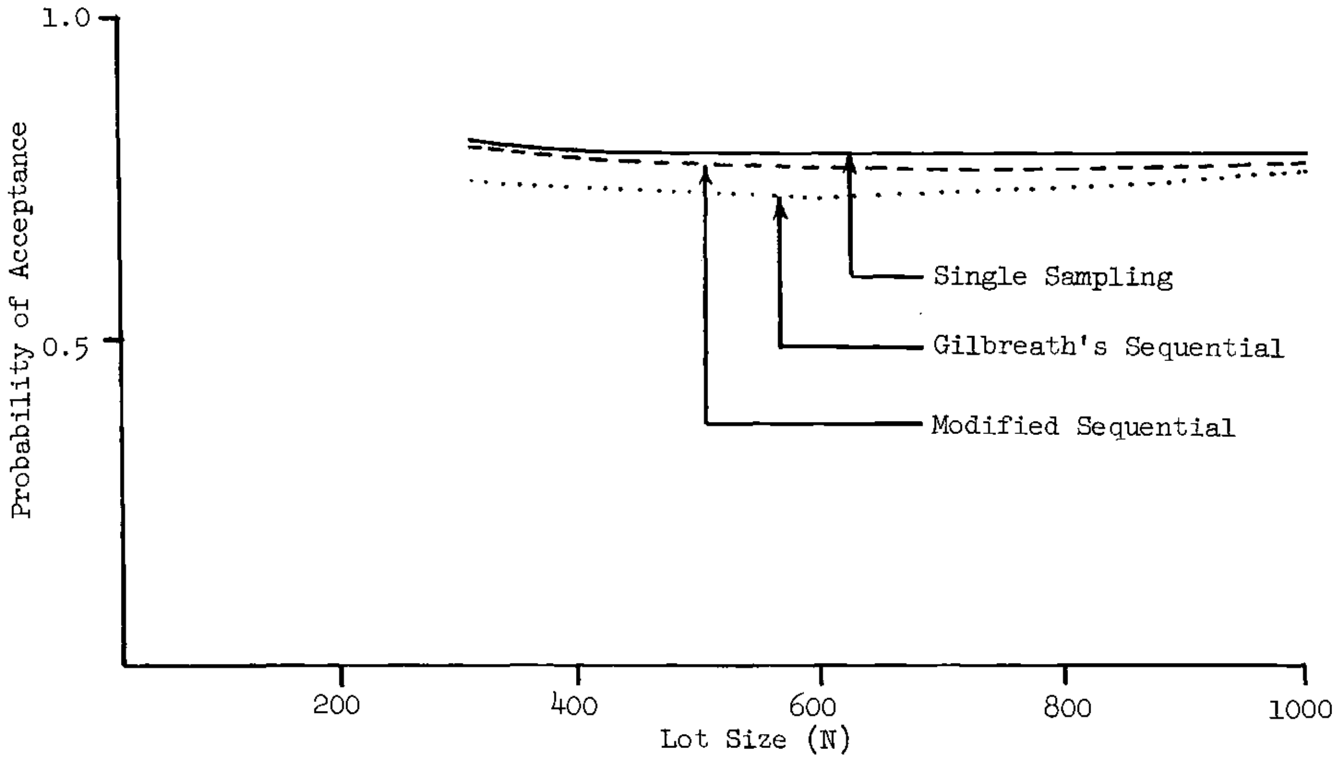

Figure 8. Comparison of Acceptance Probabilities
$p_1 = 0.02$, $p_2 = 0.10$, $K_r = K_s = 0.05$

The modified procedure is rejected without inspection when $K_r \leqq p_1$ and accepted without inspection when $K_r \geqq p_2$. For $p_1 < K_r < p_2$, the procedure accepts or rejects according to the cost analysis.

Effect of Lot Size on Sequential Plans ($p_2 - p_1 = 0.4$)

For $K_r = 0.2$ and $N = 4$, the sequential decision rule is equivalent to the single sampling plan ($n = 1$ and $c = 0$). For $N = 12$, the sequential plan is equivalent to ($n = 2$ and $c = 0$), with curtailed inspection. For lot sizes between 24 and 114, the sequential plan is equivalent to the single sampling plan ($n = 3$ and $c = 0$), with curtailed inspection. Beyond a lot size of 164, the procedure is truly sequential.

For $K_r = 0.3$, the decision rule is sequential for all values of N. The probability of acceptance increases indicating a penalty for wrongly rejecting good items. For $K_r = 0.4$, the decision rule is equivalent to single sampling ($n = 1$ and $c = 0$), when N is less than 158. The probability of acceptance again increases indicating sensitivity of design for wrongly rejecting good items.

For $p_2 - p_1 = 0.08$ and $K_r = 0.05$, the decision rule is equivalent to single sampling ($n = 8$, $c = 0$), when N is less than 304. It is truly sequential when N is more than 304.

For $p_2 - p_1 = 0.24$ and $K_r = .12$, the decision procedure is truly sequential for all values of N.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The results of this study indicate the following conclusions.

1. The modified sequential procedure is comparable to Gilbreath's sequential procedure. The average unit costs are not significantly different from Gilbreath's procedure.

2. The modification of Gilbreath's decision procedure did not help to reduce the economic gap between Hald's optimum single sampling and Gilbreath's sequential sampling.

3. Average sample sizes are of the same order in both Gilbreath's sequential procedure and the modified sequential procedure. Lots are sentenced after examination of small samples, which is the reason for increased costs in the sequential procedures.

4. The modified sequential procedure is economically superior to accepting lots without inspection. Gilbreath's sequential procedure gives the same result.

Recommendations

On the basis of this research, the following recommendations are made.

1. A further modification in Gilbreath's decision rule is recommended which is as follows: the inspection will be terminated at any

point, for which the expected decision loss is less than the cost of inspecting one more item plus the expected decision loss if the lot is sentenced at that point, and less than the cost of inspecting two more items plus the expected decision loss if the lot is sentenced at that point, and also less than the cost of inspecting three more items plus the expected decision loss if the lot is sentenced at that point. Otherwise, one additional item will be inspected and the procedure continued until it is economical to sentence the lot with no further inspection.

2. Research towards developing an analytical solution for evaluating the expected cost of Gilbreath's sequential procedure is recommended. A dynamic programming approach may be appropriate for this purpose.

3. Gilbreath recommended that the avoidance of early decisions resulting from the inconclusiveness of the information available from very small samples would improve the proposed sequential procedure. In sequential probability ratio tests, sample size is constrained, i.e., a minimum is imposed by operating characteristic. Development of a constraint on sample sizes with these Bayesian sequential procedures might improve their cost relative to Hald's optimum single sampling. One constraint on sample size may be to continue sequential procedure to a point where sequential sample size equals to Hald's optimum single sample size.

APPENDICES

APPENDIX I

ALGOL PROGRAM FOR THE MODIFIED SEQUENTIAL PROCEDURE

```
COMMENT  SUSHIL KUMAR IE 700 MIXED BINOMIAL SEQENTIAL SAMPLING;
REAL  E,EP,G,R,S,F,A,H,HP,J,JP,K,KP,L,LP,Q,QP,T,TP,U,UP,C,B,RN,
      KPA,KPB,KPC,KPD,KPE,KPF,KPG,TPA,TPB,TPC,TPD,TPE,TPF,TPG,
      TPH,TPI,TPJ,KW,KWP,LOT,S1,S2,DD,TH,TL;
INTEGER  AB,BC,M,XS,SS,ACCEPT,D,O,XL,V,RNP,OP,Z,ZO,I,CC,LL;
REAL ARRAY  W[0:100],P[0:10],X[0:1022],N[0:100];
LABEL  L1,L2,L3,L4,L5,L6,L7,L8,L9,L10,L11,L13,Y,Y1;
FILE IN  SKIN (2,10);
FILE OUT  SKOUT  6 (2,15);
FORMAT  F1(//,X10,"N=",I4,"   LOT=",F4.2,"   XL=",I4,"   SS=",
        I4,"   XS=",I4,"   ACCEPT=",I2,"   KW=",F7.5),
        F2(//,X10,"VALUE OF N EXCEEDS 10000"),
        F3(//,X10,"NO DECISION","N=",I4,"LOT=",F4.2,"XL=",I4,
        "SS=",I4),
        F4(//,X10,"KWP=",F8.4),
        F5(X10,"   A=",F7.5,"   KP=",F7.5,"   KPG=",F7.5),
        F6(X10,"   R=",F7.5,"   S =",F7.5,"    F =",F7.5);
LIST   LST1(N[BC],LOT,XL,SS,XS,ACCEPT,KW),
       LST2(N[BC],LOT,XL,SS);
REAL PROCEDURE   MIN(S1,S2);
REAL   S1,S2;
  BEGIN
  IF S1>S2  THEN MIN←S2 ELSE MIN←S1 ;
  END OF PROCEDURE MIN ;
REAL PROCEDURE RR ;
  BEGIN
  DOUBLE (CC,0,DD,0,×,←,TH,TL);
  DOUBLE (TH,TL,ENTIER(TH),0,-,←,DD,TL) ;
  RR←DD ;
  END ;
WRITE   SKOUT[NO] ;
READ ( SKIN,/,AB,R,S,F,M) ;
FOR I←1 STEP 1 UNTIL M DO
READ ( SKIN,/,W[I] ) ;
FOR I←1 STEP 1 UNTIL M DO
READ ( SKIN,/,P[I] ) ;
WRITE ( SKOUT,F6,R,S,F ) ;
```

```
CC←549755813885 ;
DD←CC/8*13 ;
FOR BC←1 STEP 1 UNTIL AB DO
READ (SKIN,/,N[BC]);
CLOSE(SKIN,RELEASE);
FOR BC←1 STEP 1 UNTIL AB DO
     BEGIN
L5  ;ACCEPT←0 ;
     E←0 ;
     H←0 ;
     J←0 ;
     JP←0 ;
     V←0 ;
     KWP←0 ;
     KPA←0 ;
     KPB←0 ;
     KPC←0 ;
     KPD←0 ;
     KPE←0 ;
     KPF←0 ;
     KPG←0 ;
     FOR I←1 STEP 1 UNTIL M DO
     E←E+W[I]×P[I] ;
     EP←N[BC]×E;
     G←R×N[BC]×(1-E);
     A←MIN(EP,G) ;
     FOR I←1 STEP 1 UNTIL M DO
     H←H+W[I]×P[I]*2 ;
     HP←(E-H)×R×(N[BC]-1);
     FOR I←1 STEP 1 UNTIL M DO
     J←J+W[I]×(1-P[I]);
     FOR I←1 STEP 1 UNTIL M DO
     JP←JP+(W[I]×(1-P[I])×P[I])/J;
     K←JP×(N[BC]-1)×(1-E)+F+S ;
     KP←HP+K;
     FOR I←1 STEP 1 UNTIL M DO
     KPA←KPA+W[I]×((1-P[I])*2) ;
```

```
     FOR I←1 STEP 1 UNTIL M DO
     KPB←KPB+(W[I]×((1-P[I]*2)×P[I])/KPA ;
     KPB←((N[BC]-2)×KPB×(1-E)×(1-(N[BC]/(N[BC]-1))×E)) ;
     FOR I←1 STEP 1 UNTIL M DO
     KPC←KPC+W[I]×P[I]×(1-P[I]);
     FOR I←1 STEP 1 UNTIL M DO
     KPD←KPD+(W[I]×P[I]×P[I]×(1-P[I]))/KPC ;
     KPD←R×((N[BC]-1)-(N[BC]-2)×KPD)×2×(1-E)×(N[BC]/(N[BC]-1))×E;
     FOR I←1 STEP 1 UNTIL M DO
     KPE←KPE+W[I]×P[I]×P[I] ;
     FOR I←1 STEP 1 UNTIL M DO
     KPF←KPF+(W[I]×(P[I]*3))/KPE;
     KPF←R×(N[BC]-2)×(1-KPF)×(N[BC]/(N[BC]-1))×E×(E-1/N[BC]) ;
     KPG←F+2×S+KPB+KPD+KPF ;
L1  :ACCEPT←0 ;
     DP←0 ;
     SS←0 ;
     XS←0 ;
     XL←0 ;
     D←0 ;
     V←V+1 ;
     FOR LL←1 STEP 1 UNTIL 30 DO
     RN←RR ;
     IF RN<0.2 THEN GO TO L2 ;
L6  :D←D+1 ;
     RN←RR ;
     IF RN≤P[1] THEN X[D]←1 ELSE X[D]←0 ;
     IF D<N[BC] THEN GO TO L6 ;
     FOR Z←1 STEP 1 UNTIL N[BC] DO
     XL←XL+X[Z] ;
     LOT←P[1] ;
     GO TO L3 ;
L2  :D←D+1 ;
     RN←RR ;
     IF RN≤P[2] THEN X[D]←1 ELSE X[D]←0 ;
     IF D<N[BC] THEN GO TO L2 ;
     FOR Z←1 STEP 1 UNTIL N[BC] DO
```

```
      XL←XL+X[Z] ;
      LOT←P[2] ;
L3   :IF A≤KP AND A≤XPG THEN GO TO L10 ELSE
      GO TO L11 ;
L11  :RN←RR ;
      IF N[BC]<10 THEN RNP←ENTIER(RN×10) ELSE
      IF N[BC]≥10 AND N[BC]<100 THEN RNP←ENTIER(RN×100) ELSE
      IF N[BC]≥100 AND N[BC]<1000 THEN RNP←ENTIER(RN×1000) ELSE
      IF N[BC]≥1000 AND N[BC]<10000 THEN RNP←ENTIER(RN×10000)
      ELSE GO TO L13; IF RNP=0 THEN GO TO L3 ELSE
      IF RNP>N[BC] THEN GO TO L3 ELSE ZO←RNP ;
      IF ZO←DP THEN GO TO L3 ELSE DP←ZO ;
      D←X[ZO] ;
      L←0 ;
      LP←0 ;
      T←0 ;
      TP←0 ;
      U←0 ;
      UP←0 ;
      TPA←0 ;
      TPB←0 ;
      TPC←0 ;
      TPD←0 ;
      TPE←0 ;
      TPF←0 ;
      TPG←0 ;
      SS←SS+1 ;
      XS←XS+D ;
      FOR I←1 STEP 1 UNTIL M DO
      L←L+(W[I]×(P[I]*XS)×(1-P[I])*(SS-XS));
      FOR I←1 STEP 1 UNTIL M DO
      LP←LP+(W[I]×(P[I]*(XS+1))×((1-P[I])*(SS-XS)))/L;
      Q←LP×(N[BC]-SS) ;
      QP←R×(N[BC]-SS)×(1-LP) ;
      B←MIN(Q,QP) ;
      FOR I←1 STEP 1 UNTIL M DO
      T←T+W[I]×(P[I]*XS)×((1-P[I])*(SS+1-XS));
```

```
      FOR I←1 STEP 1 UNTIL M DO
      TP←TP+(W[I]×(P[I]*(XS+1))×((1-P[I])*(SS+1-XS)))/T ;
      FOR I←1 STEP 1 UNTIL M DO
      U←U+W[I]×(P[I]*(XS+1))×((1-P[I)*(SS-XS));
      FOR I←1 STEP 1 UNTIL M DO
      UP←UP+(W[I]×(P[I]*(XS+2))×((1-P[I])*(SS-XS)))/U;
      C←S+(N[BC]-SS-1)×((TP×(1-LP))+(R×(1-UP)×LP));
      FOR I←1 STEP 1 UNTIL M DO
      TPA←TPA+(W[I]×(P[I]*(XS)))×((1-P[I])*(SS+2-XS));
      FOR I←1 STEP 1 UNTIL M DO
      TPB←TPB+(W[I]×(P[I]*(XS+1))×((1-P[I])*(SS+2-XS)))/TPA;
      FOR I←1 STEP 1 UNTIL M DO
      TPC←TPC+(W[I]×(P[I]*(SX+1))×((1-P[I])*(SS+1-XS)));
      FOR I←1 STEP 1 UNTIL M DO
      TPD←TPD+(W[I]×(P[I]*(XS+2))×((1-P[I])*(SS+1-XS)))/TPC;
      FOR I←1 STEP 1 UNTIL M DO
      TPE←TPE+(W[I]×(P[I]*(XS+2))×((1-P[I])*(SS-XS)));
      FOR I←1 STEP 1 UNTIL M DO
      TPF←TPF+(W[I]×(P[I]*(XS+3))×((1-P[I])*(SS-XS)))/TPE;
      TPG←2×S+((N[BC]-SS-2)×TPB)×(1-((N[BC]-SS-2)/(N[BC]-SS))
          ×TPB)×(1-((N[BC]-SS-2)/(N[BC]-SS-1))×TPB);
      TPH←R×((N[BC]-SS-1)-((N[BC]-SS-2)×TPD))×2×(1-((N[BC]-SS-2)/
          (N[BC]-SS))×TPD)×((N[BC]-SS-2)/(N[BC]-SS-1))×TPD;
      TPI←R×(N[BC]-SS-2)×(1-TPF)×((N[BC]-SS-2)/((N[BC]-SS)×
          (N[BC]-SS-1)))×TPF×((N[BC]-SS-2)×TPF-1));
      TPJ←TPG+TPH+TPI;
      IF B≤C AND B≤TPJ THEN GO TO L9 ELSE
      GO TO L8 ;
L8   :IF SS<(N[BC]-2) THEN GO TO L11;
      GO TO Y1 ;
L9   :IF B=QP THEN GO TO Y ;
      ACCEPT←ACCEPT+1 ;
      GO TO Y ;
L10  :IF A=G THEN GO TO Y ;
      ACCEPT←ACCEPT+1 ;
Y    :KW←(S×SS+ACCEPT×(XL-XS)+(1-ACCEPT)×R×((N[BC]-SS)-(XL-XS)))/
         N[BC] ;
```

```
     KWP←KWP+KW ;
     WRITE ( SKOUT,F1,LST1);
     GO TO L4 ;
Y1  :WRITE(SKOUT,F3,LST2);
L4  :IF SS=0 THEN GO TO L7 ELSE IF V<100 THEN GO TO L1 ;
     WRITE (SKOUT,F4,KWP);
     GO TO L7 ;
L13 :WRITE(SKOUT,F2);
L7  :END ;
END.
```

APPENDIX II

GLOSSARY OF SYMBOLS

N = Lot size

n = Sample size

x = Cumulative number of defective items in accumulated sample

y = The number of defectives in the uninspected portion of the lot

C_a = Loss accompanying acceptance of a defective item

C_r = Loss resulting from rejection of a good item

C_s = Variable sampling cost per item inspected

C_f = Fixed sampling cost

K_f = Ratio of the fixed sampling cost to the decision loss accompanying acceptance of a defective item

K_s = Ratio of the variable sampling cost per item inspected to the decision loss accompanying acceptance of a defective item

K_r = Ratio of the decision loss resulting from rejection of a good item to the decision loss accompanying acceptance of a defective item

$E(y|x_n)$ = The expected number of defectives in the uninspected portion of the lot given x defectives in a sample of size n drawn from the lot

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